Exm: A non-simple left primitive ring.

D div. ring, V_D vector space, $R := \mathrm{End}(V_D)$ ($\cong$ column-finite matrices)

Then R acts on V from the left.

${}_RV$ is simple: If $v,w \in V\setminus\{0\}$, $\exists \varphi \in R$ s.t. $\varphi(v) = w$ [extend v to basis]

${}_RV$ is faithful: If $\varphi \in R$ is s.t. $\forall v \in V: \varphi(v) = 0 \Rightarrow \varphi = 0$.

So $\mathrm{ann}({}_RV) = \underline{0}$.

$\Rightarrow R$ is left primitive.

If $\dim V_D = \infty$, then R is not simple:

$I := \{\varphi \in R : \dim(\mathrm{im}\,\varphi) < \infty\} \underset{\neq}{\trianglelefteq} R$ ($\cong$ matrices w. finitely many nonzero rows).

$\oslash$

Def: Let R, S be rings. An **(R, S)-bimodule** is a set M

that is
- a left R-module
- a right S-module
- and $\forall r \in R\, \forall s \in S\, \forall m \in M: \quad (rm)s = r(ms)$

Exm:
- If R is commutative, every R-module is an (R,R)-bimodule
- For arbitrary rings R: Let $M \in \text{Mod-}R$, $E := \mathrm{End}(M_R)$.

  Then M is a left E-module ($\varphi \cdot m := \varphi(m)$ for $\varphi \in E$, $m \in M$),

  and $\varphi(mr) = \varphi(m)r$, so M is an (E,R)-bimodule.
- If ${}_RM \in R\text{-Mod}$, $E := \mathrm{End}({}_RM)$, then ${}_RM_{E^{op}}$ is a

  (R, E^{op})-bimodule: $\quad m \cdot (\varphi \circ_{op} \psi) = (\varphi \circ_{op} \psi)(m) = (\psi \circ \varphi)(m) = \psi(\varphi(m))$

$= (m\cdot\varphi)\cdot\psi$

Alternatively, view M as left R-, left E-module subject to the actions commuding: $\varphi(rm) = r\varphi(m)$.

•) ⚠ If $M_R \in \text{Mod-}R$, then ${}_{R^{op}}M \in R^{op}\text{-Mod}$ via $r\cdot m := mr$

In general this is not a bimodule structure!

[Suppose $(am)b = a(mb)$
$$\begin{matrix} (am)b & = & a(mb) \\ \| & & \nparallel \\ mab & & mba \end{matrix}$$
differ i.g., e.g. if $\operatorname{ann}(m)=0$ and $ab \neq ba$]

4.1 The Jacobson Density Theorem

Def: Let M be an (R,S)-bimodule, $E = \operatorname{End}(M_S)$. Then R acts densely on M_S if $\forall \varphi\in E\ \forall m_1,\dots,m_k \in M\ \exists r\in R\ \forall i:\ rm_i = \varphi(m_i)$.

(note $R\to E$, $r\mapsto (m\mapsto rm)$ is a ring hom.)

Lemma 4.2 If D is a div. ring and V an (R,D)-bimodule, then R acts densely on V_D $\Longleftrightarrow$

finite tuple!

$\forall$ lin. independent $(v_1,\dots,v_k)\in V^k\ \forall (v_1',\dots,v_k')\in V^k\ \exists r\in R\ \forall i:\ rv_i = v_i'$

(Proof clear)

Cor 4.3 If $\dim V_D = n < \infty$, then $\operatorname{End}(V_D) \cong M_n(D)$ is the only subring of $\operatorname{End}(V_D)$ that acts densely.

Proof: Trivially $E := \operatorname{End}(V_D)$ acts densely. Fix a D-basis $(e_1,\dots,e_n)$ of V.

Suppose $R\subseteq E$ acts densely and $\varphi\in E$.

$\Rightarrow \exists \psi\in R\ \forall i\in\{1,\dots,n\}:\ \varphi(e_i) = \psi(e_i) \Rightarrow \varphi = \psi\in R$. $\square$

Thm 4.4 (Jacobson Density Theorem, also: Chevalley-Jacobson Density Theorem)

Let R be a ring, ${}_RM$ a semisimple left R-module, $S = \operatorname{End}({}_RM)^{op}$

Then R acts densely on M_S.

Lemma 4.5 ${}_RM \in \text{Mod-}R$ semisimple, $S = \operatorname{End}({}_RM)^{op}$, $E := \operatorname{End}(M_S)$

Then every R-submodule of M is an E-submodule (& conversely)

Proof: E-submodules are R-submodules because $R \to E$, $r \mapsto (m \mapsto rm)$

is a ring hom.

Let ${}_RN \leq {}_RM$ be an R-submodule, and ${}_RN'$ s.t. ${}_RM = {}_RN \oplus {}_RN'$ [T2.12(c)]

Define $\pi: {}_RM \to {}_RM$ s.t. $\pi|_N = id$, $\pi|_{N'} = 0$ (projection on N along N').

$\Rightarrow \pi \in S$.

E acts on M from the left: $\varphi \cdot m = \varphi(m)$, ${}_EM_S$ is a bimodule

$$\forall \varphi \in E \; \forall n \in N: \quad \varphi(n) = \varphi(\pi(n)) = \varphi(n \cdot \pi) \underset{\substack{\uparrow \\ \text{bimodule structure}}}{=} \varphi(n) \cdot \pi = \pi(\varphi(n))$$

So $\varphi(N) \subseteq \pi(M) = N$

$\Rightarrow N$ is an E-submodule. $\square$

Proof of Thm 4.4: $E := \operatorname{End}(M_S)$. Let $\varphi \in E$, $m_1, \dots, m_k \in M$

Show: $\exists r \in R \; \forall i: \; \varphi(m_i) = r m_i$.

${}_R\tilde{M} := {}_RM^k$ (semisimple)

$$\tilde{S} := \operatorname{End}({}_R\tilde{M}) = \operatorname{End}({}_RM^k) \cong \overset{\swarrow \text{matrix ring}}{M_k}(\operatorname{End}(M_R)^{op}) = M_k(S)$$

Define $\tilde{\varphi}: \tilde{M} \to \tilde{M}$, $(x_1, \dots, x_k) \mapsto (\varphi(x_1), \dots, \varphi(x_k))$

<u>Claim</u>: $\tilde{\varphi} \in End(\tilde{M}_{\tilde{S}})$

<u>Proof of Claim</u>: $\tilde{\varphi}(\underline{x} + \underline{y}) = \tilde{\varphi}(\underline{x}) + \tilde{\varphi}(\underline{y})$ ✓ $(\underline{x}, \underline{y} \in \tilde{M})$

Let $\underline{x} = (x_1, \dots, x_k) \in \tilde{M}$, $s = (s_{ij})_{1 \leq i,j \leq k} \in \tilde{S}$, $s_{ij} \in S$

$$\Rightarrow (\underline{x}\, s)_j = \sum_{i=1}^{k} x_i s_{ij}$$

$$\Rightarrow \tilde{\varphi}(\underline{x}\, s) = \tilde{\varphi}\left(\sum_{i=1}^{k} x_i s_{i1}, \dots, \sum_{i=1}^{k} x_i s_{ik}\right) =$$

$$= \left(\varphi\left(\sum_{i=1}^{k} x_i s_{i1}\right), \dots, \varphi\left(\sum_{i=1}^{k} x_i s_{ik}\right)\right)$$

$$= \left(\sum_{i=1}^{k} \varphi(x_i) s_{i1}, \dots, \sum_{i=1}^{k} \varphi(x_i) s_{ik}\right)$$

$$= (\varphi(x_1), \dots, \varphi(x_k))\, s = \tilde{\varphi}(\underline{x})\, s.$$ $\square$ (Claim)

Consider ${}_R\tilde{C} := R(m_1, \dots, m_k) \leq {}_R\tilde{M}$

$\overset{[L4.5]}{\Longrightarrow}$ ${}_R\tilde{C}$ is an $End(\tilde{M}_{\tilde{S}})$-submodule of ${}_R\tilde{M}$.

$\Rightarrow \tilde{\varphi}(\tilde{C}) \subseteq \tilde{C} \Rightarrow \tilde{\varphi}(m_1, \dots, m_k) = (\varphi(m_1), \dots, \varphi(m_k)) = r(m_1, \dots, m_k)$

for some $r \in R$. So $\varphi(m_i) = r m_i$ $\forall i$.

$\square$

<u>Cor 4.6</u>: If additionally M_S is f.g., then $R \to End(M_S)$

is surjective.

Proof: Let $M_S = \langle m_1, \dots, m_k \rangle_S$. Let $\varphi \in E := \mathrm{End}(M_S)$

Let $r \in R$ s.t. $\varphi(m_i) = r m_i$ $\forall i$ [T4.4].

If $m \in M$, then $m = \sum_{i=1}^{k} m_i s_i$ with $s_i \in S$

$$\Rightarrow \varphi(m) = \varphi\Big(\sum_{i=1}^{k} m_i s_i\Big) = \sum_{i=1}^{k} \varphi(m_i) s_i = \sum_{i=1}^{k} (r m_i) s_i \overset{(R,S)\text{-bimodule}}{=} \sum_{i=1}^{k} r(m_i s_i) = rm.$$

$\square$

Most important case: ${}_R M$ simple $\Rightarrow$ $S = D$ division ring, M_D D-vector space, and $R \to \mathrm{End}(M_D)$ maps R onto a dense ring of linear operators on a vector space (here $D = \mathrm{End}({}_R M)^{op}$).

Thm 4.7 (Structure Theorem for Left Primitive Rings)

Let R be left primitive with faithful simple left R-module V and $D := \mathrm{End}({}_R V)^{op}$. Then R embeds as a dense ring of linear operators, $R \hookrightarrow \mathrm{End}(V_D)$, on the right D-vector space V.

Further:

(i) if R is left artinian, then $n := \dim V_D < \infty$ and $R \cong M_n(D)$

(ii) if R is not left artinian, then $\dim V_D = \infty$.

For all $n \geq 1$, there is a subring $R_n \subseteq R$ and a surjective ring hom. $R_n \to M_n(D)$.

Remark: (i) recovers T2.21 (c) $\Rightarrow$ (e) with a different proof.

Proof: Since $\operatorname{ann}({}_R V) = \underline{0}$, $R \hookrightarrow \operatorname{End}(V_D)$ via left multiplication.

R acts densely by T4.4.

Case 1: $n := \dim V_D < \infty$.

Then $R \hookrightarrow \operatorname{End}(V_D)$ is surjective [C4.6], so $R \cong \operatorname{End}(V_D) \cong M_n(D)$.

In particular, R is artinian.

Case 2: $\dim V_D = \infty$. Choose D-lin. indep. $v_1, v_2, \ldots$

Let $W_n := \langle v_1, \ldots, v_n \rangle_D$, $R_n := \{ r \in R : rW_n \subseteq W_n \}$,

$I_n := \{ r \in R : rW_n = \underline{0} \}$

$\Rightarrow R_n \subseteq R$ is a subring, $I_n \trianglelefteq R_n$

$R_n W_n \subseteq W_n \Rightarrow W_n$ is an R_n-module and $I_n = \operatorname{ann}({}_{R_n} W_n)$

$\Rightarrow \varphi_n : R_n \to \operatorname{End}(W_{n,D}) \cong M_n(D)$ is a ring hom w. $\ker \varphi_n = I_n$.

For any $w_1, \ldots, w_n \in W$, $\exists r \in R : rv_i = w_i$ [T4.4]

$r \in R_n$

$\Rightarrow \varphi_n$ surjective.

Claim: R is not left artinian

I_n is a left ideal of R.

$\exists r \in R : rv_1 = \ldots = rv_n = 0$, $rv_{n+1} \neq 0$ $\Rightarrow r \in I_n \setminus I_{n+1}$

$\Rightarrow I_1 \supsetneq I_2 \supsetneq \ldots$ $\square$

Cor 4.8 R left primitive $\Leftrightarrow$ R isomorphic to a dense subring of linear operators on a right vector space over a division ring.

4.2 Application: Jacobson's Commutativity Theorem

(After Herstein, Bell)

Thm 4.9 Let R be a ring. Suppose $\forall x \in R\ \exists n = n(x) > 1 : \ x^n = x$.
Then R is commutative.

In a boolean algebra, always $x^2 = x$. Easy: $\forall x \in R,\ x^2 = x \Rightarrow R$ comm.:

$1 = (-1)^2 = -1$, $\quad (x+y)^2 = \underbrace{x^2}_{=x} + xy + yx + \underbrace{y^2}_{=y} = x + y \Rightarrow xy + yx = 0$

$\overset{-1=1}{\Longrightarrow} \underline{xy = yx}$.

What if $x^3 = x$?

Note: Every idempotent ($e^2 = e$) is central

[Since $0 = e(1-e) = (1-e)e$: $\quad \forall r \in R$:

$er(1-e) = [er(1-e)]^3 = 0 \Rightarrow er = ere$

$(1-e)re = [(1-e)re]^3 = 0 \Rightarrow re = ere$]

- $(r^2)^2 = r^4 = r^2 \Rightarrow \forall r \in R$: r^2 idempotent $\Rightarrow r^2 \in Z(R)$
- $1 + r = (1+r)^3 = 1 + 3r + 3r^2 + \underbrace{r^3}_{=r} \Rightarrow 3r = -3r^2 \in Z(R)$
- $(1+r)^2, r^2 \in Z(R) \Rightarrow (1+r)^2 - 1 - r^2 = 2r \in Z(R)$
- $\Rightarrow r = 3r - 2r \in Z(R)$

Lemma 4.10 If R is as in T 4.9, then $J(R) = 0$

Proof: Let $x \in J(R)$, $n > 1$ s.t. $x^n = x$

$\Rightarrow \underbrace{(1 - x^{n-1})}_{\in R^\times} x = 0 \Rightarrow x = 0$.

$\square$

So $R \hookrightarrow \prod\limits_{I \text{ left primitive}} R/I$, each R/I retains the assumption and

it suffices to prove [T4.9] for left primitive R.

<u>Lemma 4.11</u> If R is as in T4.9, and left primitive, then R

is a division ring.

<u>Proof.</u> By [T4.7], $R \hookrightarrow End(\underset{\text{faithful simple R-module}}{V_D})$ densely w. D division ring.

We show $\dim V_D = 1$, then $R \cong End(D_D) = D$.

Suppose $v, w \in V$ are D-linearly independent.

[T4.4]

$\Longrightarrow \exists r \in R: \quad rv = w, \quad rw = 0$

$\Longrightarrow \forall n > 1: \quad r^n v = 0. \quad \lightning \quad r^{n(r)} = r$

$\square$

<u>Lemma 4.12</u> (Herstein) Let D be a division ring w. $\operatorname{char} D = p > 0$.

Suppose $a \in D \setminus Z(D)$ s.t. $a^{p^n} = a$ for some $n \geq 1$.

$\Rightarrow \exists x \in D, \ i > 1: \quad a^i \neq a$ and $xax^{-1} = a^i$

<u>Thm 4.13</u> (Wedderburn's "Little" Theorem) Every finite division ring is a field.

(Proofs later)

<u>Proof of T4.9:</u> By L4.10, L4.11, wlog $R = D$ is a division ring.

In D, $(2 \cdot 1_D)^n = 2 \cdot 1_D$ for some $n > 1$, so $\operatorname{char} D \neq 0$.

$\Rightarrow \operatorname{char} D = p$ for some prime p, wlog $\mathbb{F}_p \subseteq D$.

Suppose there exists $a \in D \setminus \mathbb{F}_p$. Then $\mathbb{F}_p[a] \subseteq D$ is a finite field (since $a^{n(a)} - a = 0$) of char $p \Longrightarrow \exists n \geq 1 : a^{p^n} = a$.

Let $x \in D$, $i > 1$ s.t. $a^i \neq a$, $x a x^{-1} = a^i$ [L4.12]

Let $m > 1$ s.t. $x^m = x$.

Then $R := {}_{\mathbb{F}_p}\langle a^k x^\ell : 0 \leq k \leq p^{n-1}, 0 \leq \ell < m \rangle$ is a finite subring of D

$\left[(a^k x^\ell)(a^{k'} x^{\ell'}) = a^k (x^\ell a^{k'} x^{-\ell}) x^{\ell+\ell'} = a^{k+k' i^\ell} x^{\ell+\ell'} \in R \right]$,

Hence a division ring $\overset{T4.13}{\Longrightarrow}$ D field $\Rightarrow$ x, a commute $\lightning$. $\square$